# Exam: Advanced Quantum Mechanics

*Thursday, July 22, 2023*

*The total number of points is 50. Do not forget to write your name and the student number on the first sheet. See that all pages are numbered. Good luck.*

1. **Translation operator:** The translation operator $\hat{T}_a$ transforms the state $|x\rangle$ into the state $|x+a\rangle$,

$$\hat{T}_a|x\rangle = |x+a\rangle, \tag{1}$$

where $|x\rangle$ is an eigenstate of the coordinate operator: $\hat{x}|x\rangle = x|x\rangle$.

   (a) Calculate the commutator $\left[\hat{x}, \hat{T}_a\right]$. [5 points]

   *Hint: What is the result of the action of this commutator on the state $|x\rangle$?*

   (b) The translation by a distance $a$ transforms a state $|\Psi\rangle$ into $|\Psi'\rangle$:

$$|\Psi'\rangle = \hat{T}_a|\Psi\rangle.$$

   Find a relation between the wave functions $\psi'(x)$ and $\psi(x)$ of these states in coordinate representation. [5 points]

   *Hint: The translation operator preserves the norm and, hence, is a unitary operator.*

   (c) Prove

$$\hat{T}_a = e^{-i\frac{a}{\hbar}\hat{p}}, \tag{2}$$

   where $\hat{p}$ is the momentum operator. [8 points]

   *Hint: Show that $\hat{T}_a$ defined by Eq.(2) transforms $|x\rangle$ into $|x+a\rangle$. Use the completeness relation for the eigenstates of the momentum operator. Alternatively, prove that $e^{-i\frac{a}{\hbar}\hat{p}}$ transforms $\hat{x}$ into $\hat{x}+a$ and show that this equivalent to $\hat{T}_a|x\rangle = |x+a\rangle$.*

2. **Coherent state:** Find the root-mean-square deviation of energy, $\Delta E$, in the coherent state of harmonic oscillator. [8 points]

   *Hint: $\Delta E = \sqrt{\langle \hat{H}^2\rangle - \langle \hat{H}\rangle^2}$. This problem can be solved without the explicit expression for the coherent state. Use the definition of this state and commutation relation for the creation and annihilation operators.*

3. **Motion in magnetic field:** An electron moves in a magnetic field described by the vector potential $\boldsymbol{A} = (-\frac{By}{2}, \frac{Bx}{2}, 0)$, where $B$ is a constant (the scalar potential $\phi$ is zero).

   (a) Show that the corresponding magnetic field vector is $\boldsymbol{B} = (0,0,B)$. [3 points]

   (b) Show that $\hat{p}_z$ and $\hat{L}_z$ are conserved. [7 points]

   *Hint: The easiest way to prove the conservation of the z-component of the orbital momentum is to separate the Hamiltonian into three terms proportional to $B^0, B^1$ and $B^2$, and consider their symmetry properties.*

   (c) Show that the electron rotates in the $xy$ plane:

$$\begin{cases} \dot{\hat{\mathrm{v}}}_\mathrm{x} = -\omega \hat{\mathrm{v}}_y, \\ \dot{\hat{\mathrm{v}}}_\mathrm{y} = +\omega \hat{\mathrm{v}}_x, \end{cases} \tag{3}$$

   with the cyclotron frequency $\omega = \frac{eB}{mc}$, where $\hat{v} = \frac{\hat{p} + \frac{e}{c}\hat{A}}{m}$ is the velocity operator. [7 points]

   *Hint: Use the Heisenberg equation of motion.*

4. **Angular momentum operator**

   Calculate the root-mean-square deviation of the $x$-component of the angular momentum operator in the state $|jm\rangle$: $\Delta j_x = \sqrt{\langle jm| \left(\hat{j}_x\right)^2 |jm\rangle - \langle jm|\hat{j}_x|jm\rangle^2}$. [7 points]

   *Hint: The use of matrix elements of $\hat{j}_\pm$ can be avoided, if one considers transformation properties of $\hat{j}_x$ and $|jm\rangle$ under rotations around the z axis.*

# Useful formulae

- **Commutators**

$$[\hat{A},\hat{B}] = -[\hat{B},\hat{A}], \quad [\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$$

- **Transformation of states and operators**

$$|\Psi'\rangle = \hat{U}|\Psi\rangle, \quad \hat{A}' = \hat{U}^\dagger\hat{A}\hat{U}, \quad \hat{U}^\dagger\hat{U} = 1$$

$$e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}} = \hat{B} + \frac{\lambda^1}{1!}[\hat{A},\hat{B}] + \frac{\lambda^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + \ldots.$$

- **Completeness relations**

$$\sum_n |n\rangle\langle n| = \int dx|x\rangle\langle x| = \int \frac{dp}{2\pi\hbar}|p\rangle\langle p| = 1$$

- **Harmonic oscillator**

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \quad [\hat{a},\hat{a}^\dagger] = 1$$

$$\begin{cases} \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger) \\ \hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}-\hat{a}^\dagger) \end{cases}$$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad \hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle$$

- **Coherent state**

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \quad \hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

- **Levi-Civita tensor**

$$\varepsilon_{nmk}\varepsilon_{kpq} = \delta_{np}\delta_{mq} - \delta_{nq}\delta_{mp}, \quad \varepsilon_{nmk}\varepsilon_{nml} = 2\delta_{kl}$$

$$\varepsilon_{ijk}A_jB_k = [\boldsymbol{A}\times\boldsymbol{B}]_i, \quad \varepsilon_{ijk}A_iB_jC_k = (\boldsymbol{A}\cdot[\boldsymbol{B}\times\boldsymbol{C}])$$

- **Particle in electromagnetic fields**

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t} - \nabla\phi, \quad \boldsymbol{B} = [\nabla\times\boldsymbol{A}], \quad \hat{H} = \frac{(\hat{\boldsymbol{p}} - \frac{q}{c}\hat{\boldsymbol{A}})^2}{2m} - q\hat{\phi}$$

- **Matrix elements of $\hat{j}_\pm = \hat{j}_x \pm i\hat{j}_y$**

$$\hat{j}_+|jm\rangle = \sqrt{(j-m)(j+m+1)}|j,m+1\rangle, \; \hat{j}_-|jm\rangle = \sqrt{(j+m)(j-m+1)}|j,m-1\rangle$$